Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel International Advanced Level

Centre Number

Candidate Number

Sample Assessment Materials for first teaching September 2018

(Time: 1 hour 30 minutes) | Paper Reference **WFM01/01**

Mathematics

International Advanced Subsidiary/Advanced Level
Further Pure Mathematics FP1

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 75.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over

S59759A

1/1/1/1/1/1/

S 5 9 7 5 9 A 0 1 3 2

Pearson

Leave blank

Answer ALL questions. Write your answers in the spaces provided.

1. Use the standard results for $\sum_{r=1}^{n} r$ and for $\sum_{r=1}^{n} r^3$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} r(r^2 - 3) = \frac{n}{4}(n+a)(n+b)(n+c)$$

where a, b and c are integers to be found.

(4)

$\sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3r$

can split summation into 2.

$\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} 3r$

factor 3 out summation

$= \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$

Summations

$\sum_{r=1}^{n} 1 = n$
$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ } must learn!

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ } in the formulae booklet

$= \frac{1}{4}n^2(n+1)^2 - 3\left[\frac{1}{2}n(n+1)\right]$

$= \frac{1}{4}n^2(n+1)^2 - \frac{3}{2}n(n+1)$

factor out $\frac{1}{4}n(n+1)$

$= \frac{1}{4}n(n+1)\left[n(n+1) - 6\right]$

$= \frac{1}{4}n(n+1)\left[n^2 + n - 6\right]$

$= \frac{1}{4}n(n+1)(n+3)(n-2)$

$= \boxed{\frac{n}{4}(n-2)(n+1)(n+3), \quad a=-2 \quad b=1 \quad c=3}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

2. A parabola P has cartesian equation $y^2 = 28x$. The point S is the focus of the parabola P.

(a) Write down the coordinates of the point S.

(1)

Points A and B lie on the parabola P. The line AB is parallel to the directrix of P and cuts the x-axis at the midpoint of OS, where O is the origin.

(b) Find the exact area of triangle ABS.

(4)

a. $y^2 = 28x$

$y^2 = 4(7)x \qquad a = 7$

focus : $(7, 0)$

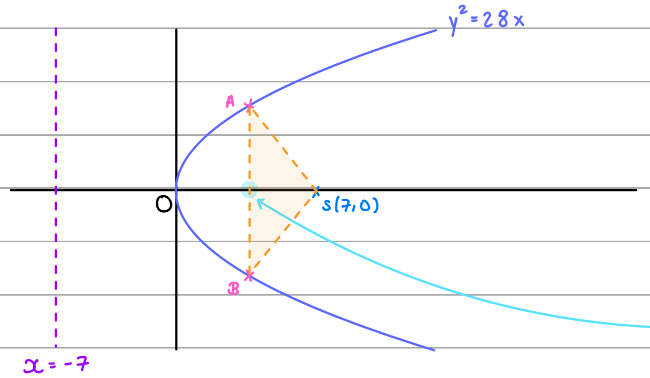
$\boxed{\text{Point } S : (7, 0)}$

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$y^2=4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$xy=c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e<1$ $b^2=a^2(1-e^2)$	$e=1$	$e>1$ $b^2=a^2(e^2-1)$	$e=\sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x=\pm\frac{a}{e}$	$x=-a$	$x=\pm\frac{a}{e}$	$x+y=\pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a}=\pm\frac{y}{b}$	$x=0, y=0$

b. Directrix of P : $x = -7$

Sketch parabola and directrix, then label points : A, B, S.



midpoint of OS : $\left(\frac{0+7}{2}, \frac{0+0}{2}\right)$

$= \left(\frac{7}{2}, 0\right)$

$\therefore$ line AB crosses through point $\left(\frac{7}{2}, 0\right)$

we know since line AB crosses through $\left(\frac{7}{2}, 0\right)$, both A and B have x-coord : $\frac{7}{2}$.

sub $x = \frac{7}{2}$ into parabola eq^n to get y-coords.

$y^2 = 28\left(\frac{7}{2}\right)$

$y^2 = 98$

$y = \pm\sqrt{98} = \pm 7\sqrt{2}$

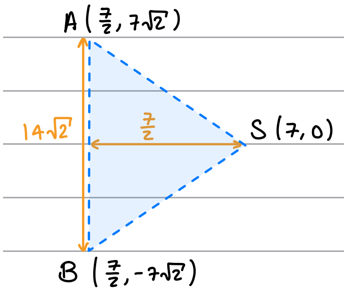
DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

Question 2 continued

Graphically, y-coord of A is +ve. $\therefore$ $A\left(\frac{7}{2}, 7\sqrt{2}\right)$

y-coord of B is -ve. $\therefore$ $B\left(\frac{7}{2}, -7\sqrt{2}\right)$



$\triangle ABS$: $\frac{1}{2} \times 14\sqrt{2} \times \frac{7}{2}$

$= \frac{49\sqrt{2}}{2}$

$\boxed{\text{Area of triangle } ABS = \frac{49\sqrt{2}}{2} \text{ units}^2}$

Q2

(Total for Question 2 is 5 marks)

Leave blank

3.

$$f(x) = x^2 + \frac{3}{x} - 1, \qquad x < 0$$

The only real root, α, of the equation $f(x) = 0$ lies in the interval $[-2, -1]$.

(a) Taking -1.5 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 2 decimal places.

(5)

(b) Show that your answer to part (a) gives α correct to 2 decimal places.

(2)

a. $x_1 = -1.5$

we are looking for 2^{nd} approximation (x_2)

Numerical Sol^{n}s of Eq^{n}s:

Newton-Raphson iteration for solving $f(x)=0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

From Newton-Raphson iteration: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

(1) must find $f(x_1)$ by subbing in x_1 (-1.5) into $f(x)$

(2) must find $f'(x_1)$ by differentiating $f(x)$ and subbing in x_1 (-1.5) into $f'(x)$.

(1) $f(-1.5) = (-1.5)^2 + \frac{3}{(-1.5)} - 1 = \frac{-3}{4}$

(2) $f(x) = x^2 + \frac{3}{x} - 1$

rewrite as indices:

$f(x) = x^2 + 3x^{-1} - 1$

$f'(x) = 2x - 3x^{-2}$

$f'(-1.5) = 2(-1.5) - 3(-1.5)^{-2} = \frac{-13}{3}$

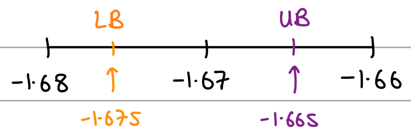
Question 3 continued

$$X_2 = (-1.5) - \frac{\left(-\frac{3}{4}\right)}{\left(-\frac{13}{3}\right)}$$

$$X_2 = -\frac{87}{52}$$

$$\boxed{\alpha = -1.67 \ (2\text{d.p.})}$$

b. find upper and lower bound of $\alpha = -1.67$ to 2 d.p.



Find $F(LB)$ and $F(UB)$

$$f(-1.675) = (-1.675)^2 + \frac{3}{(-1.675)} - 1 = 0.01458022388 \quad (LB)$$

$$f(-1.665) = (-1.665)^2 + \frac{3}{(-1.665)} - 1 = -0.02957 68018 \quad (UB)$$

There is a sign change from positive to negative and $f(x)$ is continuous, $\therefore$ a root is $\alpha = 1.67$ (2d.p.)

Q3

(Total for Question 3 is 7 marks)

Leave blank

Leave blank

4. Given that

$$\mathbf{A} = \begin{pmatrix} k & 3 \\ -1 & k+2 \end{pmatrix}, \text{ where } k \text{ is a constant}$$

(a) show that $\det(\mathbf{A}) > 0$ for all real values of k,

(3)

(b) find $\mathbf{A}^{-1}$ in terms of k.

(2)

a. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(x) = (a)(d) - (b)(c)$

$\det(A) = (k)(k+2) - (3)(-1)$

$= k^2 + 2k - (-3)$

$= k^2 + 2k + 3$

$\det(A) = k^2 + 2k + 3$

$k^2 + 2k + 3$

$= (k+1)^2 - 1 + 3$ complete the square

$= (k+1)^2 + 2$

(for all values of k, positive or negative, $(k+1)^2 + 2 > 0$ always

$\therefore \det(A) > 0$ for all real values of k.

b. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X^{-1} = \dfrac{1}{\det(x)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$-b = -(3) = -3$

$-c = -(-1) = 1$

$$A^{-1} = \frac{1}{k^2+2k+3} \begin{bmatrix} k+2 & -3 \\ 1 & k \end{bmatrix}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Leave blank

5.

$$2z + z^* = \frac{3 + 4\mathrm{i}}{7 + \mathrm{i}}$$

Find z, giving your answer in the form $a + b\mathrm{i}$, where a and b are real constants. You must show all your working.

(5)

let $z = a + bi$

z^* is the complex conjugate of z.

(flip sign of imaginary component

$z^* = a - bi$

$2z + z^* = \frac{3+4i}{7+i}$

$2(a+bi) + (a-bi) = \frac{3+4i}{7+i}$

$2a + 2bi + a - bi = \frac{3+4i}{7+i}$

$3a + bi = \frac{3+4i}{7+i}$

Rewrite $\frac{3+4i}{7+i}$ in form $a+bi$:

multiply top and bottom by complex conjugate of denominator

$\frac{3+4i}{7+i} \times \frac{7-i}{7-i} = \frac{(3+4i)(7-i)}{(7+i)(7-i)}$

$= \frac{21 - 3i + 28i - 4i^2}{49 - 7i + 7i - i^2}$

$= \frac{21 - 4(-1) + 25i}{49 - (-1)}$

$= \frac{25 + 25i}{50}$

$= \frac{\cancel{(25)}(1+i)}{\cancel{(25)}(2)}$

$= \frac{1+i}{2}$

$= \frac{1}{2} + \frac{1}{2}i$ ← in the form $a+bi$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

Question 5 continued

$3a + bi = \frac{1}{2} + \frac{1}{2}i$

Equate the real components and imaginary components:

$3a = \frac{1}{2}$ $\quad$ $bi = \frac{1}{2}i$

$a = \frac{1}{6}$ $\quad$ $b = \frac{1}{2}$

$\boxed{\therefore z = \frac{1}{6} + \frac{1}{2}i}$

Q5

(Total for Question 5 is 5 marks)

6. The rectangular hyperbola H has equation $xy = 25$

(a) Verify that, for $t \neq 0$, the point $P\left(5t, \dfrac{5}{t}\right)$ is a general point on H. **(1)**

The point A on H has parameter $t = \dfrac{1}{2}$

(b) Show that the normal to H at the point A has equation

$$8y - 2x - 75 = 0$$

(5)

This normal at A meets H again at the point B.

(c) Find the coordinates of B. **(4)**

a. sub in $P\left(5t, \frac{5}{t}\right)$ $x = 5t$, $y = \frac{5}{t}$ into $xy = 25$ eq^n.

$xy = 25$

$(5\cancel{t})\left(\frac{5}{\cancel{t}}\right) = 25$

$\boxed{\therefore \text{ point P is a general point on H.}}$

b. sub in $t = \frac{1}{2}$ into Point $P\left(5t, \frac{5}{t}\right)$ to find coordinates of point A.

$A: \left(5\left(\frac{1}{2}\right), \frac{5}{\left(\frac{1}{2}\right)}\right)$

$A: \left(\frac{5}{2}, 10\right)$

Differentiate rectangular hyperbola, H w.r.t. x:

$xy = 25$

$y = \frac{25}{x} = 25x^{-1}$

$\frac{dy}{dx} = -25x^{-2} = \frac{-25}{x^2}$

sub in coord. A $\left(\frac{5}{2}, 10\right)$:

$\left.\frac{dy}{dx}\right|_{\left(\frac{5}{2}, 10\right)} \frac{-25}{\left(\frac{5}{2}\right)^2} = -4$

$M_{tangent} = -4$

$M_{normal} = \frac{1}{4}$

Leave blank

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

Question 6 continued

set up line eq^n $y-y_1 = m(x-x_1)$ with $m = 1/4$

$(x_1, y_1) = \left(\frac{5}{2}, 10\right)$

$y - 10 = \frac{1}{4}\left(x - \frac{5}{2}\right)$ } multiply both sides by 4.

$4(y-10) = x - \frac{5}{2}$

$4y - 40 = x - \frac{5}{2}$ } multiply both sides by 2.

$2(4y-40) = 2\left(x - \frac{5}{2}\right)$

$8y - 80 = 2x - 5$

$\boxed{8y - 2x - 75 = 0 \text{ // shown.}}$

c. (1) normal eq^n: $8y - 2x - 75 = 0$

(2) rectangular hyperbola eq^n: $xy = 25$

solve these 2 eq^{n}s simultaneously to find points of intersection (coord. A and B)

(2) $xy = 25$ } multiply both sides by 2

$2(xy) = 2(25)$

$2xy = 50$

(1) $8y - 2x - 75 = 0$ } rearrange for $2x$

$8y - 75 = 2x$

$(8y - 75)y = 50$ sub in (1) into (2).

$8y^2 - 75y = 50$

$8y^2 - 75y - 50 = 0$

$(y - 10)(8y + 5) = 0$

$y = 10$ or $\underline{\underline{y = -\frac{5}{8}}}$

↑

coord. A.

sub into either eq^n (1) or (2) to find x-coord

$x\left(-\frac{5}{8}\right) = 25$

$x = 25 \div -\frac{5}{8}$

$x = -40$

$\boxed{\therefore B = \left(-40, -\frac{5}{8}\right)}$

Leave blank

7.

$$\mathbf{P} = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **P**. **(3)**

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line with equation $y = x$

(b) Write down the matrix **Q**. **(1)**

Given that the transformation V followed by the transformation U is the transformation T, which is represented by the matrix **R**,

(c) find the matrix **R**. **(2)**

refers to matrix R

(d) Show that there is a value of k for which the transformation T maps each point on the straight line $y = kx$ onto itself, and state the value of k. **(4)**

a. $\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ ← Anticlockwise rotation (Given in Formulae booklet) through ϑ about O

$$\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{bmatrix}$$

Equate the entries in the matrix: $\cos\vartheta = \frac{5}{13}$ $\quad \sin\vartheta = \frac{12}{13}$

$$\frac{\sin\vartheta}{\cos\vartheta} = \frac{\frac{12}{\cancel{13}}}{\frac{5}{\cancel{13}}} = \frac{12}{5}$$

$$\tan\vartheta = \frac{12}{5}$$

$$\vartheta = \tan^{-1}\left(\frac{12}{5}\right)$$

$$\vartheta = 67.38013505^\circ$$

$\therefore$ Matrix P represents an anticlockwise rotation through 67.4° (3s.f.) about O.

DO NOT WRITE IN THIS AREA

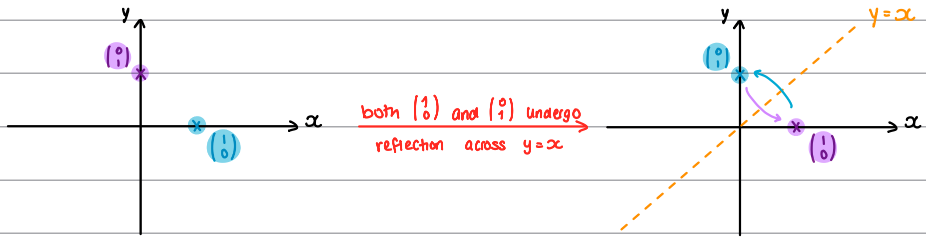
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 7 continued

b. Start by drawing a coordinate axis and mark coordinates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

then transform both of these coordinates by transformation (reflection in line $y = x$)



identity matrix I

rewrite matrix but now replace coordinates with their transformations in this specific order

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{new matrix represents transformation } Q} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

c. order : $T = UV$

matrix : $R = PQ$

$$R = \begin{bmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} \left(\frac{5}{13}\right)(0) + \left(-\frac{12}{13}\right)(1) \\ \left(\frac{12}{13}\right)(1) + \left(\frac{5}{13}\right)(0) \end{bmatrix}$$

$$\boxed{R = \begin{bmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}}$$

d. $$\begin{bmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix} \begin{bmatrix} x \\ Kx \end{bmatrix} = \begin{bmatrix} x \\ Kx \end{bmatrix}$$

$$\begin{bmatrix} \left(-\frac{12}{13}\right)(x) + \left(\frac{5}{13}\right)(kx) \\ \left(\frac{5}{13}\right)(x) + \left(\frac{12}{13}\right)(kx) \end{bmatrix} = \begin{bmatrix} x \\ Kx \end{bmatrix}$$

Leave blank

Question 7 continued

(1) $-\frac{12}{13}x + \frac{5}{13}kx = x$

(2) $\frac{5}{13}x + \frac{12}{13}kx = kx$

(1) $-\frac{25}{13}x + \frac{5}{13}kx = 0$

$\frac{1}{13}x(-25 + 5k) = 0$

$\frac{1}{13}x = 0 \qquad -25 + 5k = 0$

$5k = 25$

$k = 5$

(2) $\frac{5}{13}x - \frac{1}{13}kx = 0$

$\frac{1}{13}x(5 - k) = 0$

$\frac{1}{13}x = 0 \qquad 5 - k = 0$

$k = 5$

Both eq^{n}s confirm $k = 5$,

$\boxed{\therefore k = 5}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Leave blank

8.

$$f(z) = z^4 + 6z^3 + 76z^2 + az + b$$

where a and b are real constants.

Given that $-3 + 8i$ is a complex root of the equation $f(z) = 0$

(a) write down another complex root of this equation. **(1)**

(b) Hence, or otherwise, find the other roots of the equation $f(z) = 0$ **(6)**

(c) Show on a single Argand diagram all four roots of the equation $f(z) = 0$ **(2)**

a. For a quartic eqⁿ with form: $ax^4 + bx^3 + cx^2 + dx + e$, there are 4 roots.
There are 3 possibilities: 4 real roots
2 real roots and 2 complex roots (conjugate pair)
4 complex roots (2 conjugate pairs).

The other complex root must be the complex conjugate of $(-3+8i)$
(flip sign of imaginary component

$\boxed{-3-8i}$

b. $(z-(-3+8i))(z-(-3-8i))(pz^2+qz+r)$
$(z+3-8i)(z+3+8i)(pz^2+qz+r)$
$(z^2+3z+\cancel{8iz}+3z+9+\cancel{24i}-\cancel{8iz}-\cancel{24i}-64i^2)(pz^2+qz+r)$
$(z^2+6z+9-64(-1))(pz^2+qz+r)$
$(z^2+6z+73)(pz^2+qz+r)$
$pz^4+qz^3+rz^2+6pz^3+6qz^2+6rz+73pz^2+73qz+73r$
$(p)z^4+(q+6p)z^3+(r+6q+73p)z^2+(6r+73q)z+(73r)$

equate coefficients with $f(z)$: (1) $p=1$
(2) $q+6p=6$
(3) $r+6q+73p=76$
(4) $6r+73q=a$
(5) $73r=b$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

Question 8 continued

(2) $q + 6p = 6$

$q + 6(1) = 6$

$q + 6 = 6$

$q = 0$

(3) $r + 6q + 73p = 76$

$r + 6(0) + 73(1) = 76$

$r + 73 = 76$

$r = 3$

quadratic factor, $z^2 + 3$

$z^2 + 3 = 0$

$z^2 = -3$

$z = \pm\sqrt{-3}$

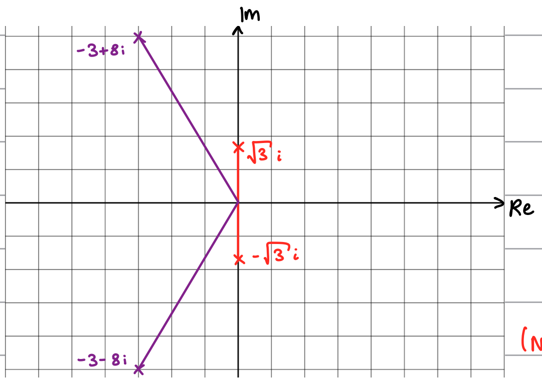
$= \pm\sqrt{(-1)(3)}$

$= \pm\sqrt{-1}\sqrt{3}$

$= \pm\sqrt{3}\,i$

$\boxed{z = \pm\sqrt{3}\,i}$

C.



(NOT DRAWN TO SCALE)

Leave blank

9. The quadratic equation

$$2x^2 + 4x - 3 = 0$$

has roots α and β.

Without solving the quadratic equation,

(a) find the exact value of

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3 + \beta^3$

(5)

(b) Find a quadratic equation which has roots $(\alpha^2 + \beta)$ and $(\beta^2 + \alpha)$, giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers.

(4)

a. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

let α, β be roots of quadratic

$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$

sum of roots : $\alpha + \beta$

product of roots: $\alpha\beta$

$2x^2 + 4x - 3 = 0$ $\div 2$

$x^2 + 2x - \frac{3}{2} = 0$

$\alpha + \beta = -2$

$\alpha\beta = -\frac{3}{2}$

i. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-2)^2 - 2\left(-\frac{3}{2}\right)$

$= 7$

$\boxed{\alpha^2 + \beta^2 = 7}$

ii. $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= (-2)^3 - 3\left(-\frac{3}{2}\right)(-2)$

$= -17$

$\boxed{\alpha^3 + \beta^3 = -17}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 9 continued

b. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - (\alpha^2 + \beta + \beta^2 + \alpha)x + ((\alpha^2+\beta) \times (\beta^2+\alpha)) = 0$

$x^2 - (\alpha^2 + \beta^2 + \alpha + \beta)x + (\alpha^2\beta^2 + \alpha^3 + \beta^3 + \alpha\beta) = 0$

$x^2 - (\alpha^2 + \beta^2 + \alpha + \beta)x + ((\alpha\beta)^2 + \alpha\beta + \alpha^3 + \beta^3) = 0$

use ans from part (ai)

use ans from part (aii)

$x^2 - ((7) + (-2))x + ((-\frac{3}{2})^2 + (-\frac{3}{2}) + (-17)) = 0$

$x^2 - 5x - \frac{65}{4} = 0$

$x^2 - 5x - \frac{65}{4} = 0$ $\times 4$

$4x^2 - 20x - 65 = 0$ ⟵ all integer coefficients

$\boxed{4x^2 - 20x - 65 = 0 \qquad a = 4, b = -20, c = -65}$

Leave blank

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

10. (i) A sequence of positive numbers is defined by

$$u_1 = 5$$
$$u_{n+1} = 3u_n + 2, \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 2 \times (3)^n - 1$$

(5)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} \frac{4r}{3^r} = 3 - \frac{(3+2n)}{3^n}$$

(6)

i. (1) Base Case: Prove true for $n = 1$

$u_1 = 2 \times (3)^1 - 1 = 5$

$u_1 = 5$

$\therefore$ true for $n = 1$

(2) Assume true for $n = k$:

$u_k = 2 \times (3)^k - 1$

(3) Prove true for $n = k + 1$:

Thinking ahead, sub in $n = k+1$ into u_n - this is what we are trying to prove using (2):

$u_{k+1} = 2 \times (3)^{k+1} - 1$

Make a note to the side so we know what we are working towards.

$u_{k+1} = 3u_k + 2$ } sub u_k assumed true from step (2)

$= 3\left[2 \times (3)^k - 1\right] + 2$

$= 3\left[2(3)^k - 1\right] + 2$ } expand 3 to the brackets

$= (2)(3)(3^k) - 3 + 2$

$= (2)(3)^1(3)^k - 1$ } index laws: $a^m \times a^n = a^{m+n}$, $3^k \times 3^1 = 3^{k+1}$

$= 2(3)^{k+1} - 1$

$= 2 \times 3^{k+1} - 1$ $\therefore$ true for $n = k+1$

(4) Conclusion

If this result is true for $n = k$, then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.

Leave blank

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Leave blank

DO NOT WRITE IN THIS AREA

Question 10 continued

ii. (1) Base Case: Prove true for $n=1$

LHS: $\sum_{r=1}^{1} \frac{4r}{3^r} = \frac{4(1)}{3^{(1)}} = \frac{4}{3}$

RHS: $3 - \frac{(3+2(1))}{3^1} = \frac{4}{3}$

LHS = RHS

$\therefore$ true for $n=1$

(2) Assume true for $n=k$:

$\sum_{r=1}^{k} \frac{4r}{3^r} = 3 - \frac{(3+2k)}{3^k}$

(3) Prove true for $n=k+1$:

Thinking ahead, sub in $n=k+1$ into RHS: this is what we are trying to prove using (2):

$\sum_{r=1}^{k+1} \frac{4r}{3^r} = 3 - \frac{3+2(k+1)}{3^{k+1}}$

Make a note so we know what we are working towards

$\sum_{r=1}^{k+1} \frac{4r}{3^r} = \sum_{r=1}^{k} \frac{4r}{3^r} + \frac{4(k+1)}{3^{k+1}}$ ← we are rewriting sum of $(k+1)$ terms as the sum of (k) terms + $(k+1)^{th}$ term - we can now use step (2).

$= 3 - \frac{(3+2k)}{3^k} + \frac{4(k+1)}{3^{k+1}}$

$= 3 - \frac{(3+2k)}{3^k} + \frac{4k+4}{3(3^k)}$

$= \frac{3(3)(3^k) - 3(3+2k) + 4k+4}{3(3^k)}$

$= \frac{3(3^{k+1}) - 9 - 6k + 4k + 4}{3^{k+1}}$

$= \frac{3(3^{k+1}) - 2k - 5}{3^{k+1}}$

$= \frac{3\cancel{(3^{k+1})}}{\cancel{(3^{k+1})}} - \frac{2k+5}{3^{k+1}}$

$= 3 - \frac{3+2k+2}{3^{k+1}}$

$= 3 - \frac{3+2(k+1)}{3^{k+1}}$ $\therefore$ true for $n=k+1$

Leave blank

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 10 continued

(4) Conclusion

If this result is true for $n = k$, then it is true for $n = k + 1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.